and so $(x \cdot y)' = e_1 x_{12} u_{23}(y_{12}u_{13} + u_{13}y_{12})u_{12}$. Now (4), (5) and (6) imply that

$$y_{12}u_{13}u_{12} = y_{12}u_{13}(e_1 + e_3)u_{12} = (\bar{y}_{23} - u_{13}y_{12})e_1u_{12} = -u_{13}y_{12}e_1u_{12} =$$

$$-u_{13}(e_1 + e_3)y_{12}e_1y_{12} = -u_{13}e_1y_{12}e_1u_{12} = 0.$$

Also $u_{23}u_{13}y_{12} = u_{23}u_{13}(e_1 + e_2)y_{12} = u_{23}u_{13}e_1y_{12} = (u_{12} - u_{13}u_{23})e_1y_{12} = u_{12}e_1y_{12}$. But then $(x \cdot y)' = e_1x_{12}u_{12}e_1y_{12}u_{12} = x'y'$.

We have proved that the mapping $x \rightarrow x'$ is a homomorphism of $\mathfrak{C}$ onto the subalgebra $\mathfrak{C}'$ of $\mathfrak{A}$ consisting of all $x'$. The kernel $\mathfrak{H}$ of this homomorphism is not $\mathfrak{C}$ since otherwise every $x' = 0$ whereas $u' = e_1u_{12}^2 = e_1(e_1 + e_2) = e_1 \neq 0$. Since $\mathfrak{C}$ is simple $\mathfrak{H} = 0$, the homomorphism is an isomorphism. This is impossible since $\mathfrak{C}$ is not associative and $\mathfrak{C}'$ is associative.

[1] See the author's "On a Certain Algebra of Quantum Mechanics," *Ann. Math.*, **35**, 65–73 (1934).

[2] For these properties see Section 18 of the author's "A Structure Theory for Jordan Algebras," *Ibid.*, **48**, 546–567 (1947).

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# *ON THE SINGULAR VALUES OF A PRODUCT OF COMPLETELY CONTINUOUS OPERATORS*

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In this note I wish to present a theorem on the singular values of a product of completely continuous operators in Hilbert space. As an application, a simple proof of a recent result of S. H. Chang will be given. The singular values of an operator $K$ are the positive square roots of the eigen-values of $K^*K$, where $K^*$ is the adjoint of $K$.

We begin with a slight generalization of a theorem of Weyl.[1]

THEOREM 1. *If $H$ is a positive, symmetric, completely continuous operator whose first $n$ eigen-values[2] are $\lambda_1, \ldots, \lambda_n$, then*

$$\det\,[(Hy_i, y_j)] \leq \lambda_1 \cdot \ldots \cdot \lambda_n \det\,[(y_i, y_j)]$$

*for any elements $y_1, \ldots, y_n$.*

Here, det $[a_{ij}]$ denotes the determinant of the $n$th order matrix with elements $a_{ij}$. Weyl's elegant proof uses an appeal to the theory of $n$-tensors. A straightforward proof may be given by using the relation $(Hy_i, y_j) = \sum_k \lambda_k (y_i, x_k)(x_k, y_j)$, where the $x_k$ form a *complete* ortho-normal set.

THEOREM 2. *If $K$ is a completely continuous operator with singular values $\alpha_i$, then det* $[(Ky_i, Ky_j)] \leq \alpha_1^2 \ldots \alpha_n^2$ *det* $[(y_i, y_j)]$.

This follows immediately from Theorem 1 if we set $H = K^*K$.

THEOREM 3. *Let $A$ and $B$ be completely continuous operators and let the singular values of $A$, $B$ and $AB$ be denoted by $\alpha_i$, $\beta_i$, $\gamma_i$, respectively. If $f$ is any function such that $f(e^x)$ is convex and increasing as a function of $x$, then for each $n$ we have* $\sum_{i=1}^n f(\gamma_i) \leq \sum_{i=1}^n f(\alpha_i\beta_i)$.

*Proof:* Let $y_1, \ldots, y_n$ form an ortho-normal set with $(AB)^*ABy_i = \gamma_i y_i$. By Theorem 2, $\gamma_1^2 \cdot \ldots \cdot \gamma_n^2 = \det [(ABy_i, ABy_j)] \leq \alpha_1^2 \cdot \ldots \cdot \alpha_n^2 \det [(By_i, By_j)] \leq \alpha_1^2 \cdot \ldots \cdot \alpha_n^2 \beta_1^2 \cdot \ldots \cdot \beta_n^2$. The result now follows by an application of a theorem of Polya.[3]

The next theorem was proved by Chang[4] using methods of function theory.

THEOREM 4. *Suppose $K = K_1 \cdot \ldots \cdot K_m$, where each $K_i$ is an operator of finite norm (integral operator with $L_2$ kernel), and let $\gamma_i$ be the singular values of $K$. Then $\sum_i \gamma_i^{2/m}$ converges.*

*Proof:* The proof is by induction on $m$. The case $m = 1$ is an immediate consequence of the definition of an operator of finite norm. Suppose the theorem holds when $K$ is a product of fewer than $m$ operators. Let $\alpha_i$ be the singular values of $K_1$, and let $\beta_i$ be the singular values of $K_2 \cdot \ldots \cdot K_m$. Using Theorem 3 and Holder's inequality, we have

$$\sum_i \gamma_i^{2/m} \leq \sum_i \alpha_i^{2/m} \beta_i^{2/m} \leq (\sum_i \alpha_i^2)^{1/m} (\sum_i \beta_i^{2/m-1})^{m-1/m}$$

In conclusion we remark that by a theorem of Chang,[4] the convergence of $\sum_i \gamma_i^{2/m}$ implies the convergence of $\sum_i |\lambda_i|^{2/m}$, where $\lambda_i$ are the eigenvalues of $K$.

[1] Weyl, H., "Inequalities Between the Two Kinds of Eigenvalues of a Linear Transformation," these PROCEEDINGS, **35**, 408–411 (1949).

[2] The eigen-values and singular values will always be arranged in order of decreasing absolute value, with repetitions according to multiplicity.

[3] Polya, G., "Remark on Weyl's Note: Inequalities Between the Two Kinds of Eigenvalues of a Linear Transformation," these PROCEEDINGS, **36**, 49–51 (1950).

[4] Chang, S. H., "On the Distribution of the Characteristic Values and Singular Values of Linear Integral Equations," *Trans. Am. Math. Soc.*, **67**, 351–368 (1949).